Proposal for a Boundary-Integral Method without Using Green's Function

NAOTO KISHI, MEMBER, IEEE, AND TAKANORI OKOSHI, FELLOW, IEEE

Abstract —A new method for solving electromagnetic boundary-value problems is presented. The new method is a modification of the conventional boundary-element method; the conventional method is modified by using the reciprocity theorem derived from Green's identity, making the use of Green's function unnecessary. To confirm the validity of the new method, numerical analyses are presented for Dirichlet- and Neumann-type boundary-value problems of a two-dimensional scalar wave equation.

I. Introduction

VARIOUS NUMERICAL methods are now available for the analysis of electromagnetic boundary-value problems [1]. In particular, since the early 1970's, integral-equation methods [2]–[7] have become popular because of the relatively short computer time required. In the past 15 years, of the integral-equation methods, the so-called boundary-element method (sometimes called the counter-integral method) has been used frequently to solve electromagnetic boundary-value problems [3]–[7].

In the boundary-element method, the boundary-integral equation is obtained first from Green's identity by using Green's function and is then solved by a discretization procedure similar to the finite-element method. As the Green's function for a two-dimensional scalar wave equation, various zero-order Bessel or Hankel functions, or their combinations, have been used [3], [4], [6], [7].

However, to formulate the boundary-integral equation, Green's function is not always necessary. The use of Green's function is sometimes even harmful because the integral equation will have a singular point when Green's function is used; the singular point makes the numerical analysis more complicated.

In this paper, we present a new boundary-integral method without using Green's function. In the new method, a homogeneous solution of the wave equation is used instead of Green's function, as has been done in [2] to solve scattering problems. We apply this method to solve Dirichlet- and Neumann-type boundary-value problems of a two-dimensional scalar wave equation. The moment method [10] is used in the computation.

Manuscript received November 28, 1986; revised June 11, 1987. This work was supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture.
The authors are with the Department of Electronic Engineering, University of Tokyo, Tokyo, Japan 113.
IEEE Log Number 8716524.

II. Formulation of Integral Equation

A. Basic Equations

Consider a two-dimensional region S surrounded by a contour Γ as shown in Fig. 1. If we denote by ϕ a scalar wave function defined in region S, ϕ satisfies the following wave equation:

$$\nabla_t^2\phi + k^2\phi = 0 \tag{1}$$

where ∇_t^2 denotes a two-dimensional Laplacian operator, and k the eigenvalue or the wavenumber of the eigenfunction ϕ.

The boundary condition is given generally as

$$p\phi + q\frac{\partial \phi}{\partial n} = 0 \tag{2}$$

where p and q are real constants. In (2), the case $p \neq 0$, $q = 0$ and the case $p = 0$, $q \neq 0$ represent Dirichlet- and Neumann-type boundary conditions, respectively. The operator $(\partial/\partial n)$ denotes the derivative in the outward normal direction on Γ.

B. Conventional Boundary-Element Method (BEM Formulation)

In conventional BEM analyses [3]–[7], the wave equation (1) is first converted to a boundary-integral equation by using the two-dimensional Green's theorem (see [3] for the derivation) as

$$\phi(\mathbf{r}') = 2\oint_\Gamma \left(G\frac{\partial \phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right) dl \tag{3}$$

where $\phi(\mathbf{r}')$ denotes the wave function at $\mathbf{r}'$ upon the contour (see Fig. 1). Function G is the two-dimensional Green's function in free space, which satisfies

$$\nabla_t^2 G + k^2 G = -\delta(\mathbf{r} - \mathbf{r}') \tag{4}$$

where δ denotes a Dirac's delta function.

In [3], [4], [6], and [7], Green's function has been chosen as

$$G(R) = \begin{cases} jH_0^{(2)}(kR)/4 & \text{(in [3], [6])} \\ C_0 J_0(kR) - Y_0(kR)/4 & \text{(in [4])} \\ -Y_0(kR)/4 & \text{(in [7])} \end{cases} \tag{5}$$

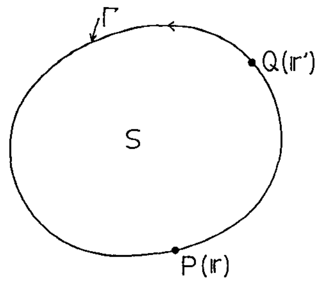


Fig. 1. A two-dimensional region S surrounded by a contour Γ.

where

$$R = |\mathbf{r} - \mathbf{r}'|$$

and

$H_0^{(2)}(kR)$	zero-order Hankel function of the second kind,
$J_0(kR)$	zero-order Bessel function of the first kind,
$Y_0(kR)$	zero-order Bessel function of the second kind,
C_0	complex constant.

In the conventional BEM, (3) is solved to obtain the solution for (1) and (2).

C. Formulation without Green's Function

We note that (3) is a boundary-integral equation; that is, it is an equation in terms of the ϕ values and their normal derivatives only on the boundary Γ. Our present concern is whether it is possible to use a homogeneous solution of (4), in other words a solution of (1), instead of G, which is the inhomogeneous solution of (4). Hereafter, we denote the homogeneous solution by ψ and call it the weight function.

If we substitute the wave function ϕ and the weight function ψ into the two-dimensional Green's theorem:

$$\int_S (\psi \nabla_t^2 \phi - \phi \nabla_t^2 \psi)\, dS = \oint_\Gamma \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) dl \qquad (6)$$

we obtain, instead of (3), the following scalar reciprocal theorem [8], [9]:

$$\oint_\Gamma \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) dl = 0 \qquad (7)$$

because both ϕ and ψ are solutions of (1).

We can solve (7) instead of (3) to obtain the solution for (1) and (2). This is the principle of the proposed method.

III. Formulation for Boundary-Element Analysis

A. Discretization of the Boundary

We apply discretization similar to that in conventional BEM analysis in solving the integral equation (7). We assume first that the region S has an elliptical or rectangular shape, with symmetries with respect to the x and y axes, and that the eigenfunction ϕ also has such symmetries. Thus, we need to consider only the first quadrant.

The boundary Γ is then approximated by N straight-line segments, which are called elements in the BEM analysis

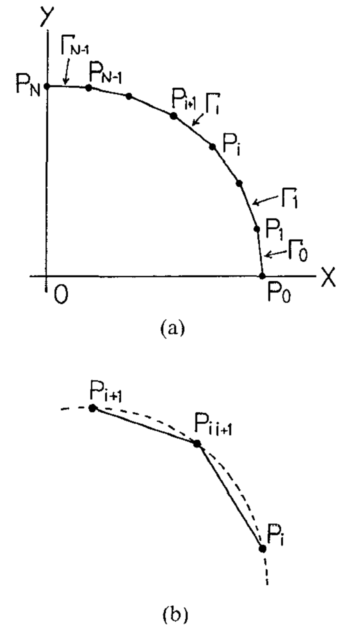


Fig. 2. Polygonal approximation of the boundary. (a) Type-1 approximation. (b) Type-2 approximation.

(see Fig. 2(a)). The function ϕ and its derivative $(\partial\phi/\partial n)$ are now defined upon the node points P_i between the two elements Γ_{i-1} and Γ_i shown in Fig. 2(a).

Another method for approximating the boundary is shown in Fig. 2(b). In this method, a segment Γ_i consists of two line elements. Therefore, the shape of the approximated boundary looks apparently the same as in Fig. 2(a) except that the number of elements N is doubled. Note, however, that in this second method the value of ϕ is defined only on old node points P_i, and not on the new inflection points P_{ii+1}. In the following, we call the two discretization methods (Fig. 2(a) and (b)) type 1 and type 2, respectively.

In both of type-1 and type-2 approximations, the value of function ϕ along element Γ_i is obtained by linear interpolation between values at node points P_i and P_{i+1} as

$$\phi = \frac{L-t}{L}\phi_i + \frac{t}{L}\phi_{i+1} \qquad (8)$$

where L denotes the length of element Γ_i in type 1 and the sum of the length of two line elements $\overline{P_i P_{ii+1}}$ and $\overline{P_{ii+1} P_{i+1}}$ in type 2, and t denotes a variable expressing the distance from the node point P_i along Γ_i. The normal derivative $(\partial\phi/\partial n)$ is also linearly interpolated in a manner similar to that in (8).

B. Weight Function

We choose the weight function ψ as circular harmonics, i.e., the product of a Bessel function of the first kind and a trigonometric function

$$\psi = J_j(kr)\cos(j\theta + \rho) \qquad (9)$$

where the order j and the phase ρ of the trigonometric part are chosen as shown in Table I so as to match the symmetry condition of the eigenfunction ϕ to be obtained.

TABLE I
SELECTION OF THE ORDER j AND PHASE ρ OF THE WEIGHT FUNCTION ψ ACCORDING TO THE SYMMETRY OF THE EIGENFUNCTION ϕ

symmetry about x or y-axis		$\Psi = J_j(kr)\cos(j\theta + \rho)$	
x-axis	y-axis	j	ρ
symmetric	symmetric	even	0
symmetric	antisymmetric	odd	0
antisymmetric	symmetric	odd	$\pi/2$
antisymmetric	antisymmetric	even	$\pi/2$

Thus, we can restrict the contour integral of (7) to the first quadrant, because the integrals along the x and y axes vanish owing to the symmetry of ϕ and its normal derivative.

C. Matrix Equation and Its Solutions

Using the above discretization and weight function, we now solve (7) by using the method of moments. We need $N+1$ linearly independent weight functions ψ, because the number of unknown values of ϕ is $N+1$ (see Fig. 2(a)). These $N+1$ ψ functions can be obtained by letting $j = 0, 1, \cdots, N$ in (9). Using these, (7) is now rewritten as

$$\sum_{i=0}^{N-1} \int_{\Gamma_i} \phi \frac{\partial \psi}{\partial n} dl = \sum_{i=0}^{N-1} \int_{\Gamma_i} \psi \frac{\partial \phi}{\partial n} dl \tag{10}$$

or, in matrix form, as

$$[A] \begin{pmatrix} \phi_0 \\ \vdots \\ \phi_N \end{pmatrix} = [B] \begin{pmatrix} \dfrac{\partial \phi_0}{\partial n} \\ \vdots \\ \dfrac{\partial \phi_N}{\partial n} \end{pmatrix} \tag{11}$$

where $[A]$ and $[B]$ are square matrices of order $(N+1)$.

Either the value of ϕ or its normal derivative $(\partial\phi/\partial n)$ is zero on the boundary in Dirichlet- and Neumann-type problems, respectively. Therefore, either the left-hand side or the right-hand side of (11) vanishes, and the eigenvalue equation is reduced to

$$\det[B] = 0 \quad \text{for a Dirichlet-boundary condition} \tag{12}$$

$$\det[A] = 0 \quad \text{for a Neumann-boundary condition.} \tag{13}$$

These equations are the equations to be solved.

Generally, however, (11) relates the values of a function and its normal derivative $(\partial\phi/\partial n)$ on the boundary. Equation (11) can be applied, therefore, not only to Dirichlet- and Neumann-type boundary-value problems but also to general electromagnetic boundary-value problems in composite media (such as dielectric waveguides) or with composite boundary conditions.

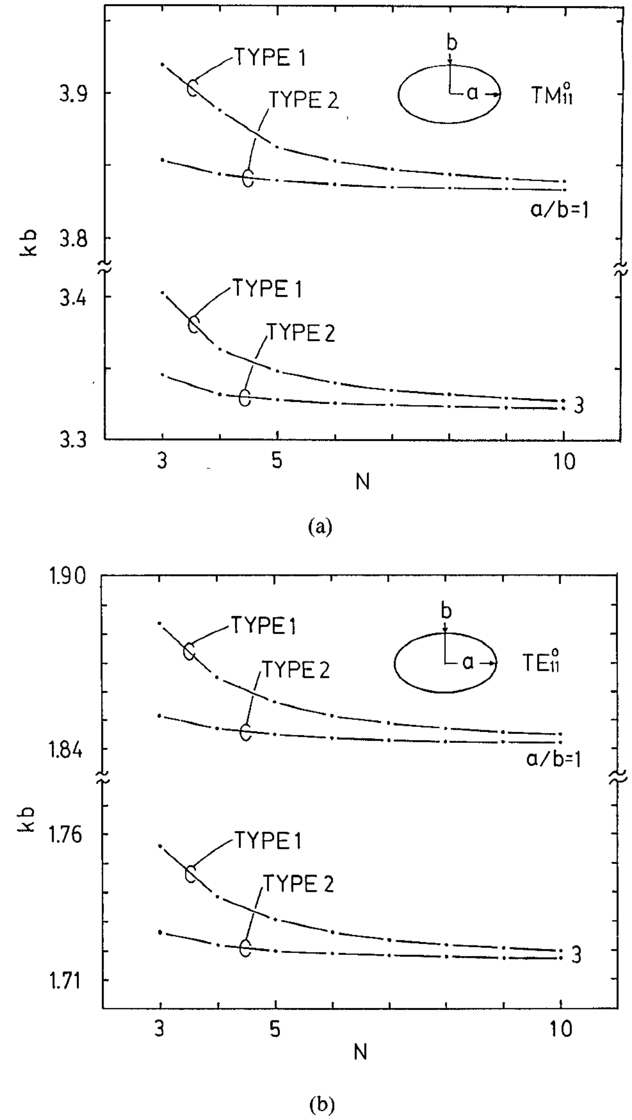


Fig. 3. Convergence characteristics for elliptical boundaries. (a) TM mode. (b) TE mode.

IV. NUMERICAL RESULTS

Some numerical solutions based upon (11)–(13) are shown in Figs. 3–7. In these figures, eigenmodes for the Dirichlet- and Neumann-type boundary conditions are called the TM and TE modes, respectively. This is because the obtained eigenfunctions ϕ give the electromagnetic field components in the direction of propagation of the TM and TE modes propagated in a metallic waveguide. The mode numbers (subscripts) are given according to those in a circular or rectangular metallic waveguide. The superscripts e and o denote, respectively, the even and odd modes with respect to the x axis, respectively.

Figs. 3 and 4 show how the calculated wavenumber kb converges as the number of elements N increases.

Fig. 3(a) and (b) shows the convergence for the elliptical boundary for the TM and TE modes, respectively. Here, TYPE 1 and TYPE 2 indicate the two methods for approximating the shape of boundary described in Section III-A.

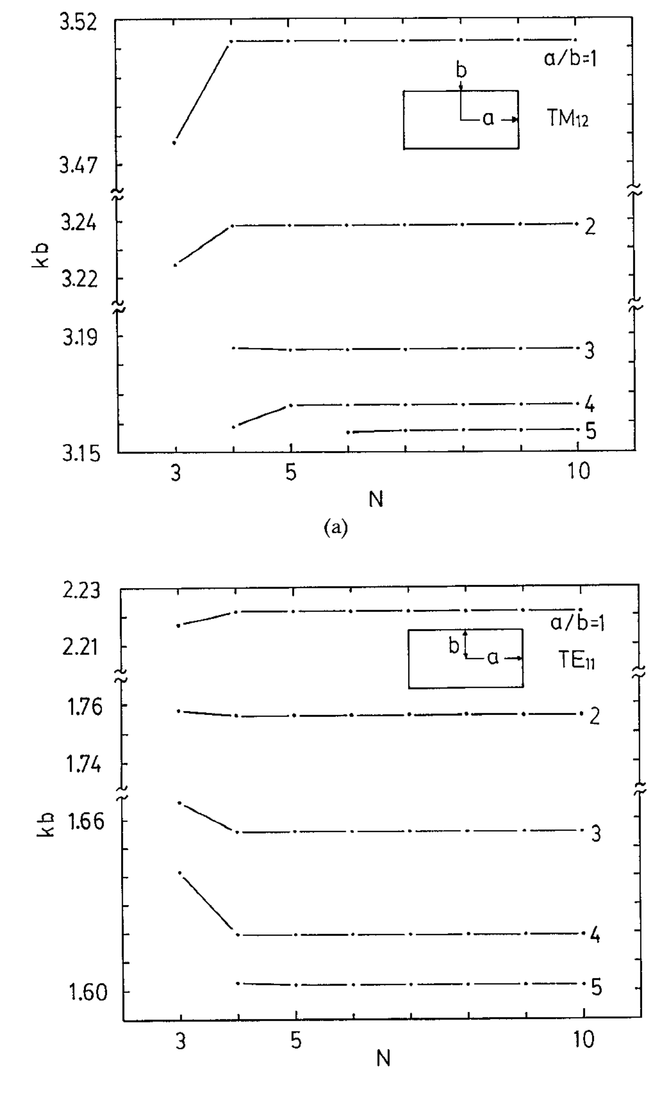


Fig. 4. Convergence characteristics for rectangular boundaries. (a) TM mode. (b) TE mode.

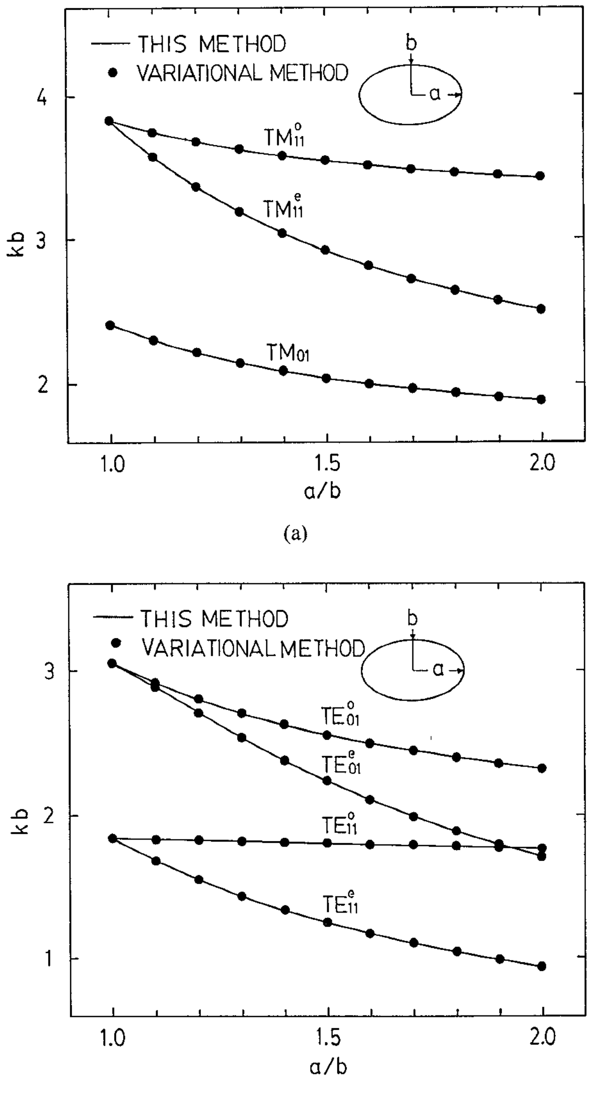


Fig. 5. Eigenvalues for elliptical boundaries as a function of the aspect ratio a/b. (a) TM mode. (b) TE mode.

When the aspect ratio a/b is unity (upper trace in Fig. 3(a) and (b)), i.e., with a circular boundary, the eigenvalues are found to converge to their analytical solutions, which are the first zeros of the Bessel functions of the first kind and their derivatives. Comparing the two types of approximations of boundary (types 1 and 2), we find that the convergence is better in type 2 than in type 1. It is also found that an eigenvalue obtained with the type-2 approximation at certain N is almost equal to that with type 1 but at twice N. This means that the shape of the boundary is more important than the node number in achieving high accuracy.

Fig. 4(a) and (b) shows the convergence for the rectangular boundary. Eigenvalues converge to analytical solutions, which are $\pi/2\sqrt{(b/a)^2+4}$ and $\pi/2\sqrt{(b/a)^2+1}$ for the TM_{12} and TE_{11} modes, respectively. An accuracy of the order of magnitude of 10^{-10} is obtained at $N=10$ independent of the aspect ratio a/b.

The solid curve in Fig. 5(a) and (b) shows the eigenvalue for the elliptical boundary as a function of the aspect ratio a/b. The results of the variational method analyses using polynomials of x and y as trial functions [11] are also shown for comparison (dots). The solid curve and the dots show good agreement.

Fig. 6 shows the difference between the eigenvalues obtained with the proposed method and the conventional BEM analyses using the integral equation (eq. (3)), both using the same number of segments N. The boundary shape is assumed as a circle and eigenmodes are TE modes. The difference is found for practical purposes to be sufficiently small and to be almost independent of the number of elements N. This means that the convergence characteristics of the proposed method are almost identical with those of the conventional BEM analysis.

Fig. 7(a) and (b) shows the values of ϕ or $(\partial\phi/\partial n)$ at nodes points upon the boundary calculated by using (11)

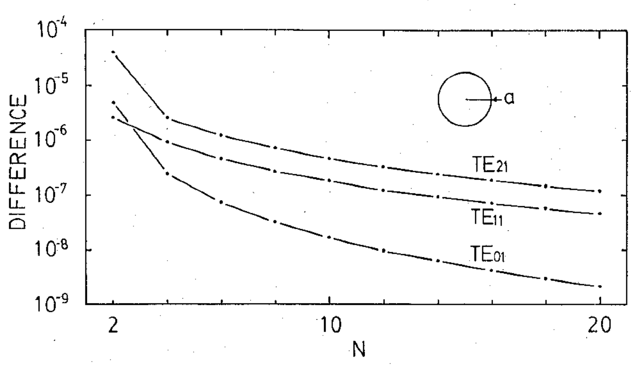
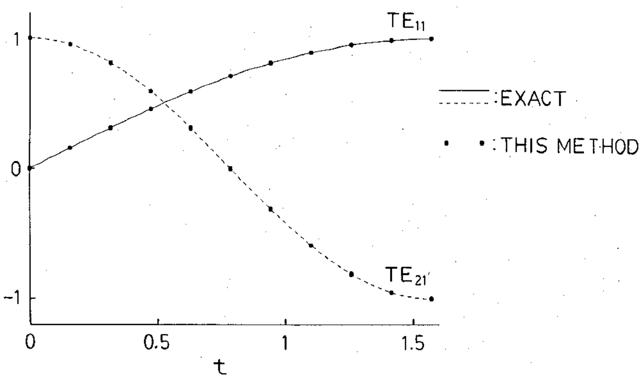
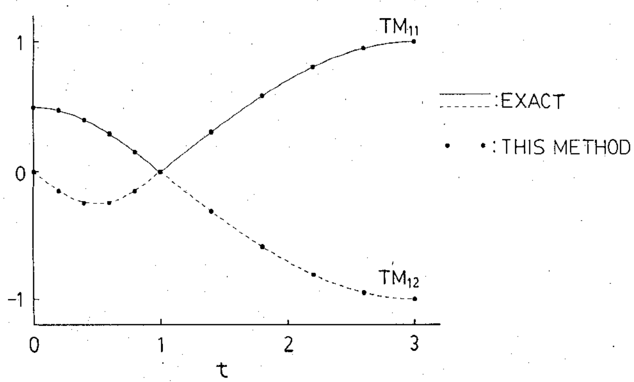


Fig. 6. Comparison of convergence characteristics with conventional BEM analysis.



(a)



(b)

Fig. 7. Values of ϕ or its normal derivatives $(\partial\phi/\partial n)$ upon the node points on the boundary. (a) ϕ of circular boundary. (b) $(\partial\phi/\partial n)$ of rectangular boundary ($a/b = 2$).

for circular and rectangular boundary shapes, respectively. The analytical solutions are also shown as solid and dotted curves. The boundary values calculated with this method show good agreement with exact ones.

V. Discussion

1) In this paper we have presented a new boundary-integral formulation without using Green's function and have applied it to boundary-element analyses of Dirichlet- and Neumann-type boundary-value problems. From the foregoing numerical results, it is found that this method is valid and useful for solving these problems.

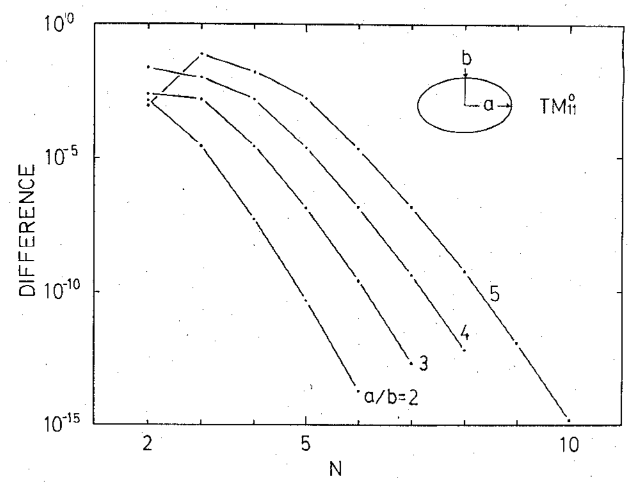


Fig. 8. Difference of the approximation of boundary between type 1 and type 2. Number of node points is that of type 2.

2) The numerical computation results shown in Figs. 3 and 4 suggest that this method can be applied to various boundary shapes with fairly rapid convergence. Comparisons with variational method and conventional BEM analyses, shown in Figs. 5 and 6, indicate that this method compares favorably with other numerical method. The result shown in Fig. 7 indicates that this method is also useful for calculating the value of the wave function on the boundary.

3) In connection with Fig. 3, it was stated that the eigenvalue obtained with the type-2 approximation at certain N is almost equal to that obtained with type-1, but at twice N. Fig. 8 shows the difference between these two eigenvalues as a function of N for the type-2 approximation. The difference is very small; therefore, we can emphasize the effectiveness of the type-2 approach, in which the approximation of the boundary shape is improved, whereas the number of node points, and hence the number of unknown variables, are kept unchanged.

4) The above statement is also supported by the fact that the convergence of the eigenvalue for the rectangular boundary (Fig. 4) is more rapid than that for the elliptical boundary (Fig. 3). We understand that this is because the approximation of the boundary shape is not necessary in the rectangular case.

5) The features of the proposed method can be summarized in comparison with the conventional BEM and other numerical methods as follows:

i) Formulation of the problem is simple and easy because Green's function is not used; hence the singular point is absent.
ii) Consideration of the modal symmetry can be done easily by proper choice of the weight function ψ.
iii) The numerical accuracy can be improved by simply improving the approximation of the boundary shape.
iv) It does not contain any spurious solutions in the numerical analysis in this paper.

VI. Conclusions

The proposed method, the boundary-integral method without Green's function, is found to be an efficient tool for solving the boundary-value problems of two-dimensional scalar wave equations.

Acknowledgment

The authors thank Dr. N. Morita of the University of Osaka and Prof. M. Hashimoto of Osaka Electro-Communications University for their helpful comments.

References

[1] S. M. Saad, "Review of numerical method for the analysis of arbitrarily shaped microwave and dielectric waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-33, pp. 894–899, Oct. 1985.

[2] N. Okamoto, "Matrix formulation of scattering by a homogeneous gyrotropic cylinder," *IEEE Trans. Antennas Propagat.*, vol. AP-18, pp. 642–649, Sept. 1970.

[3] T. Okoshi and T. Miyoshi, "The planar circuit—An approach to microwave integrated circuitry," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 245–252, Apr. 1972.

[4] Y. Ayasli, "Analysis of wide-band stripline circulators by integral equation technique," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 200–209, Mar. 1980.

[5] C. A. Brebbia, *Boundary Element Method for Engineers.* London: Pentech Press, 1978.

[6] S. Kagami and I. Fukai, "Application of boundary-element method to electromagnetic field problems," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 455–461, Apr. 1984.

[7] M. Matsubara, "Boundary element analysis of polarization holding fibers," (in Japanese), *Trans. IECE Japan*, vol. J67-B, pp. 968–973, Sept. 1984.

[8] E. A. N. Whitehead, "The theory of parallel-plate media for microwave lenses," *Proc. Inst. Elec. Eng.*, vol. 98, pt. III, pp. 133–140, Jan. 1951.

[9] T. Itoh and R. Mittra, "An analytical study of the echelette grating with application to open resonators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 319–328, June 1969.

[10] R. F. Harrington, *Field Computation by Moment Method.* New York: Macmillan, 1968.

[11] R. M. Bulley and J. B. Davies, "Computation of approximate polynomial solutions to TE modes in arbitrary shaped waveguide," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 440–446, Aug. 1969.

Naoto Kishi (S'85–M'87) was born in Tokyo, Japan, on March 11, 1960. He received the B.S. and M.S. degrees from the University of Electro-Communications, Tokyo, in 1982 and 1984, respectively, and the Ph.D. degree from the University of Tokyo in 1987, all in electrical engineering.

In 1987 he became a Research Assistant in the Department of Electronics at the University of Electro-Communications. His main fields of interest are numerical analysis methods for electromagnetic fields, microwave and lightwave circuits, and transmission lines of various types.

Dr. Kishi is a member of the Institute of Electronics, Information and Communication Engineers of Japan.

Takanori Okoshi (S'56–M'60–SM'81–F'82) was born in Tokyo, Japan, on September 16, 1932. He received the B.S., M.S., and Ph.D. degrees from the University of Tokyo in 1955, 1957, and 1960, respectively, all in electrical engineering.

In 1960 he was appointed Instructor, and in 1961, Associate Professor in the Department of Electronic Engineering, University of Tokyo. From 1963 through 1964, on leave from the University of Tokyo, he was with Bell Laboratories, Murray Hill, NJ, where he was engaged in research on electron guns. In 1972 he joined the Technical University of Munich on a temporary basis as a Guest Professor. In January 1977 he became a Professor at the University of Tokyo. At present Dr. Okoshi is also a Guest Research Fellow of the Radio Research Laboratory of Japanese Government, Commission-D Vice-Chairman of URSI (International Union of Radio Science), and President of the Japanese National Committee for URSI. His main fields of interest are optical fibers and optical fiber communications, in particular, coherent optical fiber communications and photon-counting optical communications.

Dr. Okoshi has written 13 books. The four in English are entitled *Three-Dimensional Imaging Techniques* (Academic Press, New York, 1976), *Optical Fibers* (Academic Press, 1982), *Planar Circuits* (Springer, Heidelberg, 1984), and *Coherent Optical Fiber Communications* (Reidel/KTK, to be published). He has been awarded 14 prizes from three Japanese academic institutions, and he holds 13 Japanese patents and five U.S. patents.